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Orthogonal Grid Generation in a 2D Domain via the Boundary Integral Technique

I. S. KANG

Chemical Engineering Department, POSTECH, P.O. Box 125, Pohang, 790 Korea

AND

L. G. LEAL

Department of Chemical and Nuclear Engineering, University of California, Santa Barbara, Santa Barbara, California 93106

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A new numerical scheme is proposed for the generation of an orthogonal coordinate grid in an arbitrary simply connected two-dimensional domain. The scheme is robust and non-iterative and is based on the conjunction of the familiar boundary integral technique with the covariant Laplace equation method for mapping. In the proposed scheme, two types of problems are considered: (1) Boundary correspondence is specified on two adjacent sides of the boundary, or (2) The distortion factor is specified in the product form $f(\xi, \eta) = \Pi(\xi)\, \Theta(\eta)$. © 1992 Academic Press, Inc.

1. INTRODUCTION

The generation of boundary-fitted orthogonal coordinates for a given 2D domain is a long-standing problem of theoretical and practical importance. Indeed, many methods have been proposed and many of these are reviewed in the book by Thompson *et al.* [1].

In general, there are four generic properties that would characterize an ideal mapping scheme:

(i) The availability of an existence proof for the solution of the problem posed by the given method [R1].

(ii) Adjustability of the grid spacing [R2].

(iii) Direct construction of the corresponding grid system in the *physical* domain for a given standard grid form (e.g., rectangular or square) in the *computational* domain [R3].

(iv) Robustness and *simplicity* of the solution method [R4].

So far, existence proofs have only been obtained for conformal mapping and its simple variations, more precisely, for a conformal mapping followed by *independent* nonlinear coordinate stretching in two coordinate directions. Therefore, the most obvious candidate for orthogonal grid generation methods is conformal mapping. Indeed, efficient methods for numerical construction of conformal mappings have been developed (see Fornberg [2] and references therein). Among them, the method of Symm [3] is noteworthy. He formulated the problem of conformal mapping from the 2D domain to the unit disk as an integral equation, which was then solved numerically. His scheme is powerful and accurate, but it has two significant deficiencies. One is the nonadjustability of the grid spacing [R2], which is intrinsic to conformal mapping. A second deficiency is that his method generates the image point in the *computational* domain for a given point in the *physical* domain instead of vice versa as desired [R3]. Later Menikoff and Zemach [4] again used an integral formulation, but they employed a Newton–Raphson iteration to satisfy the third requirement [R3] (i.e., to solve the direct mapping problem instead of the inverse problem).

Generally speaking, however, a more efficient approach to the direct problem is to solve the pde's for the physical space coordinate variables, rather than inverting an integral equation. One mapping scheme based upon this latter approach is due to Mobley and Stewart [5], which satisfies the requirements 1 and 3 ([R1] and [R3]) and is also reasonably flexible with regard to the requirement 2 ([R2]). Mobley and Stewart derived a set of pde's for the physical space coordinate variables in terms of computational domain coordinates that allow nonlinear stretching of coordinates from conformal coordinates. However, their formulation requires that boundary data be adjusted to correspond to a specified distortion factor, $f(\xi, \eta)$ (or vice versa), and this requires iterative solution of the mapping problem. Later, Ryskin and Leal [6] generalized Mobley and Stewart's idea by proposing the *covariant Laplace equation* method. Their scheme is ideal for the requirements [R2] and [R3] because the covariant Laplace equations

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are for the physical space variables in terms of computational domain coordinates and the grid spacing can be adjusted by choice of their distortion factor, $f(\xi, \eta)$. They proposed basically two different variations of the basic mapping technique, a *strong constraint* method that was designed primarily for free-boundary problems in which $f(\xi, \eta)$ is specified a priori, but no direct control is exerted on the placement of coordinate points along the boundaries, and a *weak constraint* method that was designed for mapping of a given fixed domain. It is the latter method that is relevant to the present discussion of mapping for a fixed domain. In the weak constraint method, the position of coordinate lines (i.e., the boundary correspondence) is specified on boundaries but the distortion factor, $f(\xi, \eta)$, is adjusted in the course of solution to achieve orthogonality. However, for this case of specified boundary correspondence on *all* boundaries, an existence proof for the mapping problem is not available ([R1]). Furthermore, the mapping problem in the weak constraint method is strongly nonlinear, and the solution method must be *iterative*, which is a disadvantage ([R4]).

In the present paper, we present a new *direct* and *noniterative* scheme based on Ryskin and Leal's *covariant Laplace equation* method, which satisfies all four basic requirements. As mentioned earlier, an existence proof is only available for limited variations of conformal mapping. Therefore, we restrict the grid adjustability in the present scheme to independent nonlinear stretching of conformal coordinates (in this case, the governing equations are basically the same as those of Mobley and Stewart). Because of the resulting restriction on the adjustability of the grid spacing, we limit our considerations to only two types of problems:

(i) The boundary correspondence between points in the mapped and physical domains is specified *arbitrarily* on two adjacent sides of the boundary.

(ii) The distortion factor $f(\xi, \eta)$ is specified in the form $f(\xi, \eta) = \Pi(\xi)\, \Phi(\eta)$.

In each case, the specified data is enough for a boundary of specified shape to determine a coordinate map from a computational domain (ξ, η) to the physical domain. However, as part of the mapping problem in case (i), we need to determine the specific form for the distortion function that is consistent with the specified boundary correspondence, as well as the boundary correspondence on the other two boundaries, while in case (ii) we need to determine the correspondence between boundary points in the mapped and physical domains that is consistent with the specified $f(\xi, \eta)$. In both cases, if we attempt to solve the mapping problem directly, this unknown data for $f(\xi, \eta)$ and/or the correspondence between mapped and physical boundary points leads to an iterative solution. In fact, Mobley and Stewart considered problems of the second type, and their solution technique was iterative.

The essence of the present mapping method is that we avoid an iterative scheme by introducing a preliminary step. This preliminary step corresponds to a conformal map and is based on the boundary integral technique (same idea as Symm [3], but in our case based on the Green's formula). As we shall see, the preliminary step allows us to determine all undetermined "parameters" of the map, i.e., the distortion function $f(\xi, \eta)$ (if not specified) and Dirichlet-type boundary conditions on all boundaries (i.e., the values taken by boundary points in the mapped domain when transformed to the physical domain). With $f(\xi, \eta)$ and the boundary correspondence specified from this initial step, the remaining mapping problem is *linear*, and we can employ a noniterative, direct scheme for its solution.

In the final section, we consider mapping problems that are intrinsically nonlinear. An example is the weak constraint method of Ryskin and Leal [6] for cases in which the boundary correspondence between the mapped and physical domains is specified on *all boundaries.* Although the preliminary conformal mapping step cannot produce a linear mapping problem, we suggest that it can still lead to a significant simplification.

2. GENERATING EQUATIONS FOR THE MAPPING

An important first step in the development of a mapping between a Cartesian and a boundary-fitted curvilinear coordinate system is to determine the equations to be satisfied by the transform functions $x(\xi, \eta)$ and $y(\xi, \eta)$ (see Fig. 1). For an orthogonal curvilinear coordinate system, these equations are the two covariant Laplace equations (Ryskin and Leal [6])

$$\frac{\partial}{\partial \xi}\left(f \frac{\partial x}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f}\frac{\partial x}{\partial \eta}\right) = 0, \tag{1a}$$

$$\frac{\partial}{\partial \xi}\left(f \frac{\partial y}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f}\frac{\partial y}{\partial \eta}\right) = 0, \tag{1b}$$

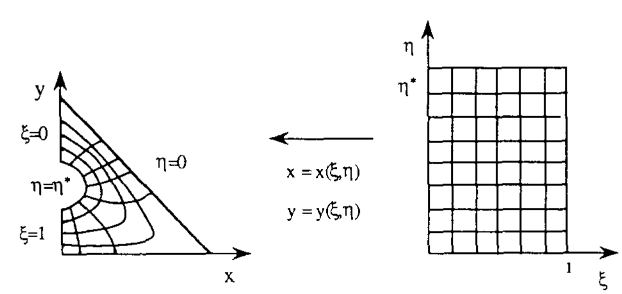


FIG. 1. Orthogonal mapping from a rectangular domain to an arbitrary simply connected 2D domain.

where $f = h_\eta / h_\xi$, and h_ξ and h_η are defined as

$$h_\xi = \sqrt{(\partial x/\partial \xi)^2 + (\partial y/\partial \xi)^2},$$
$$h_\eta = \sqrt{(\partial x/\partial \eta)^2 + (\partial y/\partial \eta)^2}.$$

In the mapping technique that is described below, this system of equations with appropriate boundary conditions is used to determine the coordinate mapping. As we can see from the definitions of f, h_ξ, and h_η, the system of Eqs. (1) is highly nonlinear, and consequently the properties of the system of equations, such as the existence and uniqueness of solutions, are not known except for some very special cases. In fact, the system (1) with boundary conditions is usually indeterminate in the sense that the solution is not unique. Since the notion of the indeterminacy of the system (1) is extremely important, we present an example in the following subsection to make the idea more concrete.

2.1. Indeterminacy of the System (1)

Here we present an example to show that the system (1) with the boundary correspondence (i.e., Dirichlet conditions for $x(\xi, \eta)$ and $y(\xi, \eta)$) and the orthogonality specified at the boundaries is not determinate, in general. As a simplest case, let us consider the problem of generating an orthogonal coordinate system in a unit square domain as shown in Fig. 2. One trivial "solution" of the mapping problem is the simple Cartesian coordinate system, $x = x(\xi, \eta) = \xi$ and $y = y(\xi, \eta) = \eta$. The question is whether

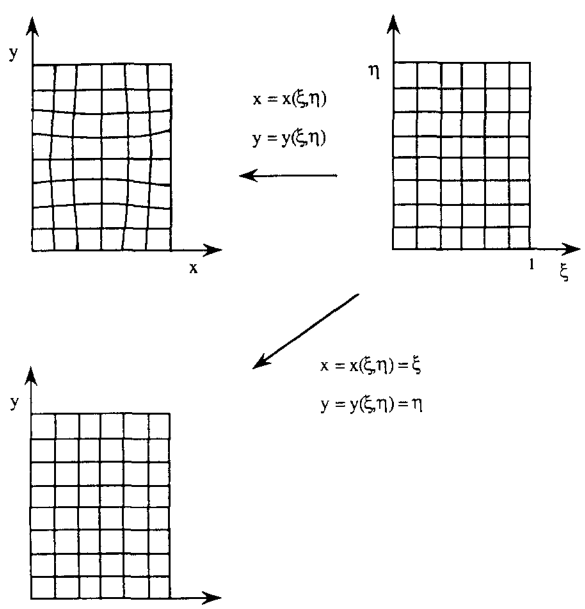


FIG. 2. Two different solutions of the system (1) which have the exactly same boundary correspondence and satisfy the orthogonality condition at the boundary.

any other orthogonal mapping exists which preserves the same boundary correspondences as the Cartesian coordinate system. To answer this question, let us begin by introducing a function $\phi(x, y) = x + V(x, y)$ in which V is an arbitrary smooth function that satisfies the following conditions:

(i) $V = V_n = 0$ at the boundaries, where the subscript n denotes differentiation in the outward normal direction.

(ii) V is antisymmetric with respect to $x = 0.5$, i.e., $V(0.5 + \delta, y) = -V(0.5 - \delta, y)$ for $0 < \delta < 0.5$.

One example for V is

$$V(x, y) = -\alpha(x - 0.5)\, x^2(x-1)^2\, y^2(y-1)^2,$$
$$0 < \alpha \ll 1.$$

Now let the level curve for ϕ starting from $(\xi, 0)$ be the coordinate line $\xi = \text{const}$. Then, the orthogonal η-coordinate is the steepest descent curve starting from $(1, \eta)$ and ending at $(0, \eta)$ because of symmetry. Obviously for all $0 < \alpha \ll 1$, the level curves and the steepest descent curves are orthogonal at any point including the boundary points. Furthermore, they have the same boundary correspondences as the Cartesian coordinate system. Since we obtain a different solution set of (1) for each arbitrary α, the system (1) clearly has *infinitely* many solutions for this specific mapping problem; i.e., the problem is not determinate.

Although we have not illustrated the indeterminacy property for more general cases, it is true for the general class of 2D domains. The most important implication is that we need more constraint(s) to make the problem determinate (i.e., the Laplace equations (1) with the boundary correspondence and orthogonality specified at all boundaries does not have a unique solution so long as $f(\xi, \eta)$ remains arbitrary).

2.2. Linear Problems of the System (1)

As suggested above, the mapping problem derived from Eq. (1) only becomes *determinate* if additional constraints are imposed. For example, if the distortion function $f(\xi, \eta)$ is specified, then the mapping equations (1a) and (1b) are linear and the mapping problem is determinate. However, this is somewhat misleading because it is *not* possible to arbitrarily specify *both* $f(\xi, \eta)$ and the correspondence of boundary points. A given distortion function is consistent only with a particular set of boundary correspondences. Hence, if $f(\xi, \eta)$ is completely specified, then the consistent set of boundary correspondences must be obtained as part of the solution of the problem and cannot be specified arbitrarily.

In the present paper, we consider the special case of mappings, where f is assumed to exist in the product form

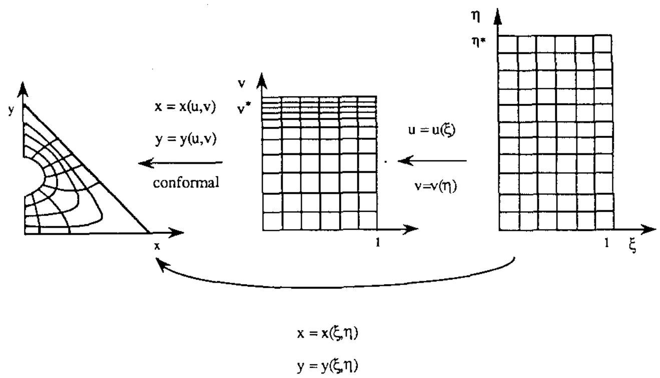


FIG. 3. Decomposition of orthogonal mapping into conformal mapping and nonlinear stretching of conformal coordinate.

$f(\xi, \eta) = \Pi(\xi)\,\Theta(\eta)$. For this class of maps, existence and uniqueness can be shown without difficulty when f is completely specified. In fact, this class of maps is just a composite consisting of a conformal map followed by independent nonlinear stretchings (or contractions) in the ξ- and η-directions as shown in Fig. 3. In this case, the distortion function is given by

$$f = \frac{h_\eta}{h_\xi} = \frac{h_v}{h_u}\frac{v'(\eta)}{u'(\xi)} = f_{\text{con}}\frac{v'(\eta)}{u'(\xi)}$$

where h_u and h_v are given as

$$h_u = \sqrt{(\partial x/\partial u)^2 + (\partial y/\partial u)^2},$$
$$h_v = \sqrt{(\partial x/\partial v)^2 + (\partial y/\partial v)^2},$$

and f_{con} denotes the distortion factor of the conformal mapping. Since the distortion factor for any conformal mapping is $f_{\text{con}} = 1$, we have

$$f = \frac{v'(\eta)}{u'(\xi)} = \frac{1}{u'(\xi)} \cdot v'(\eta) = \Pi(\xi)\,\Theta(\eta). \tag{2}$$

For this general class of maps, we will be concerned with the two distinct cases:

(1) The distortion factor f is specified in the form $f = \Pi(\xi)\,\Theta(\eta)$ without any specification of the boundary correspondences.

(2) The boundary correspondences are specified for two adjacent (i.e., not opposite) sides of the boundary, but without specification of f except for the assumption that it exists in product form.

In either case, if we try to solve the mapping problem directly, we must use an iterative method. In case (1), we must determine the particular boundary correspondence that is consistent with the specified form for f, and in a direct solution scheme this means that we would have to iterate on the boundary conditions for Eq. (1). For case (2), we must determine $f(\xi, \eta)$ (as well as the boundary correspondence on the opposite boundaries), and again, an iterative solution would seem necessary.

Here, however, we are interested in a non-iterative scheme. As mentioned earlier, the essence of our mapping method is that we avoid iteration by introducing a preliminary step. This preliminary step corresponds to a conformal map and is based on the boundary integral technique. As will be shown in Section 3, the preliminary step allows us to determine all undetermined parameters of the map, i.e., the distortion function $f(\xi, \eta)$ and the consistent Dirichlet type boundary conditions on all boundaries. Once $f(\xi, \eta)$ and/or boundary correspondences are determined from the initial step, the remaining mapping problem (Eq. (1)) is linear and can be solved non-iteratively.

2.3. Conformal mapping

As indicated above, our grid generation scheme is based on the use of conformal mapping as a preliminary step. Thus, in the present subsection, we briefly review a few known facts about conformal mapping. The celebrated Riemann mapping theorem guarantees the existence of a conformal map between any two decent simply connected domains. However, it is extremely important to note that the Riemann mapping theorem applies only to the open sets, i.e., the domains inside the boundaries, not to the closed sets. The implication is that conformality of the boundary points is neither guaranteed nor required in the theorem. In fact, conformality of the boundary points can be obtained only if the boundaries in the mapped and physical domains are conformal.

Now let us discuss the conformal mapping between a given physical domain and a *rectangular* domain with Cartesian coordinates. Such a conformal mapping can always be achieved by solving for two conjugate harmonic functions of the map in the domain Ω as shown in Fig. 4, i.e.,

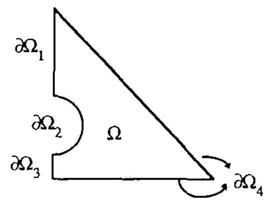
$$\nabla^2 u = 0 \quad \text{in} \quad \Omega \tag{3a}$$

with

$$\begin{pmatrix} u \\ \dfrac{\partial u}{\partial n} \\ u \\ \dfrac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} \partial\Omega_1 \\ \partial\Omega_2 \\ \partial\Omega_3 \\ \partial\Omega_4 \end{pmatrix}$$

and

$$\nabla^2 v = 0 \quad \text{in} \quad \Omega \tag{3b}$$



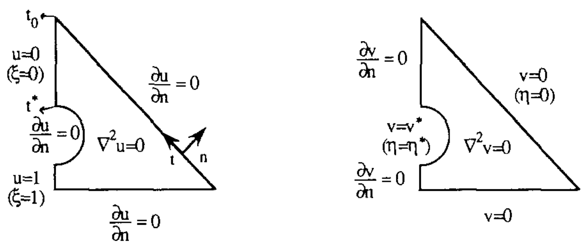


FIG. 4. Two conjugate harmonic functions for conformal mapping.

with

$$\begin{pmatrix} \dfrac{\partial v}{\partial n} \\ v \\ \dfrac{\partial v}{\partial n} \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v^* \\ 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} \partial\Omega_1 \\ \partial\Omega_2 \\ \partial\Omega_3 \\ \partial\Omega_4 \end{pmatrix}.$$

In Fig. 4, **n** is the outgoing normal vector and **t** is the tangential vector along the boundary (positive in the counterclockwise direction). As we can see in Fig. 4, at all boundary points except for some corner points, the conditions $\partial u/\partial n = 0$ or $\partial v/\partial n = 0$ ensure orthogonality. In Fig. 4, it should be noted that the value v^* for the v-problem is not free but is determined from the Cauchy–Riemann condition on $\partial\Omega_1$,

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial n}.$$

In particular,

$$v^* = -\int_{t_0}^{t^*} \left(\frac{\partial u}{\partial n}\right) dt. \tag{4}$$

As we can see in (3), the governing equations for the conformal map are linear and it is quite easy to show that the solution for each problem is unique.

So far, in the present section, we have discussed the possibility of a non-iterative grid generation scheme starting from some general properties of the mapping equations (Eq. (1)). In the following section, the overall grid generation method will be discussed in detail.

3. METHOD OF GRID GENERATION

3.1. Step 1: Preliminary Step

As suggested by the sketch in Fig. 3, the grid system for $f = \Pi(\xi)\,\Theta(\eta)$ is basically the same as for conformal mapping, except for the density of the grid distribution; i.e., the coordinate lines for $f = \Pi(\xi)\,\Theta(\eta)$ coincide with the coordinate lines of the conformal mapping, but the corresponding ξ or η values are different from the u and v values of the conformal map. Therefore the first step of our grid generation method is to determine the boundary correspondence between the physical domain and the intermediate (u, v)-domain. In this first step, we do not have to solve the whole problem defined in (3). Instead, we need only to determinee $\mathbf{u} = (u, v)$ values for the boundary points because we are interested only in the boundary correspondence at this step. For this purpose, it is natural and convenient to use a boundary integral (BI) technique for the solution of (3). The detailed implementation will be discussed in Section 4. The computed (u, v) values for the boundary can be represented as functions of the arc length along the *physical* boundary, i.e.,

$$u_b = u_b(t), \qquad v_b = v_b(t), \tag{5}$$

where the subscript b is for the boundary. Since we use a finite number of nodes for the boundary integral technique, we can use cubic spline fittings with the appropriate end point conditions (usually $u_b''(t) = 0$) to interpolate to specific values of t. These spline functions will then be used to determine u and v values corresponding to *given* points on the physical boundary. Inversely, we can develop functional relationships for (x, y) in terms of (u_b, v_b), i.e.,

$$x = x(u_b, v_b), \qquad y = y(u_b, v_b), \tag{6}$$

which can be used to determine (x, y) on the physical boundary for a given point (u_b, v_b) on the boundary in the mapped domain.

Now, we determine the Dirichlet-type boundary conditions for the covariant Laplace equations ((1a) and (1b)). In particular, we determine the correspondence between boundary points in the (ξ, η)-domain and the values of (x, y) of boundary points in the physical domain. As mentioned earlier, we are concerned with two distinct problems, and the details of this step are a bit different, depending on the specific case. Therefore, we separately discuss each type of problem.

3.1. *The case of specified* $f = \Pi(\xi)\,\Theta(\eta)$

From (3) we have

$$\frac{1}{u'(\xi)}\,v'(\eta) = \Pi(\xi)\,\Theta(\eta). \tag{7}$$

Now, since the right hand side of (7) is given, we can decompose (7) into two equations as

$$\frac{1}{u'(\xi)} = C\Pi(\xi), \qquad \text{for} \quad 0 \leqslant \xi \leqslant 1,$$

$$v'(\eta) = \frac{1}{C}\,\Theta(\eta), \qquad \text{for} \quad 0 \leqslant \eta \leqslant \eta^*,$$

where, the unknown constant C is determined from the condition $u(1) = 1$ (see Fig. 4). Therefore, we have

$$u(\xi) = \int_0^{\xi} \frac{ds}{\Pi(s)} \Big/ \int_0^1 \frac{ds}{\Pi(s)} \qquad \text{for} \quad 0 \leqslant \xi \leqslant 1 \tag{8a}$$

and

$$v(\eta) = \int_0^{\eta} \Theta(s)\, ds \Big/ \int_0^1 \frac{ds}{\Pi(s)} \qquad \text{for} \quad 0 \leqslant \eta \leqslant \eta^*. \tag{8b}$$

In (8b), the unknown upper bound of integration η^* is chosen so that $v(\eta^*) = v^*$. Equations (8a) and (8b) allow us to transform from specific boundary values in the (ξ, η) plane to corresponding values of u and v in the intermediate conformal mapping domain. Normally, the desired values for ξ and η would be chosen, as indicated in Fig. 3, by splitting the intervals $(0 \leqslant \xi \leqslant 1,\ 0 \leqslant \eta \leqslant \eta^*)$ into even parts so that coordinate lines in (ξ, η) correspond to a rectangular (Cartesian) grid.

Once (u_b, v_b) values are known, corresponding to the desired pattern of coordinate lines in (ξ, η), the corresponding boundary values for (x, y) in the physical domain are obtained via the inverted form (Eq. (6)) of the boundary-integral results for the conformal map. Thus, with $f(\xi, \eta)$ specified, and boundary values for the functions $x(\xi, \eta)$ and $y(\xi, \eta)$ specified (i.e., $\mathbf{x} = \mathbf{x}(u_b(\xi), v_b(\eta))$), the mapping problem is now reduced to the linear problem of solving (1) subject to Dirichlet boundary conditions.

3.1.b. *The Case of Specified Boundary Correspondence for Two Adjacent Sides of the Boundary*

When the boundary correspondence is specified, we mean that specific (ξ, η) values are assigned to specific points (x, y) on the boundaries of the physical domain. For the case in which the boundary correspondence is given on two adjacent boundaries, we follow the following steps:

(s1) Solve the BI conformal mapping problem. Then, we know from Eq. (5) that a point at a distance t around the physical boundary corresponds to a certain point (u, v) in the intermediate domain. Thus, each (ξ, η) pair that has been associated with a boundary point in the physical domain is now associated with a point (u, v).

(s2) On the *opposite* boundaries where the boundary correspondence is not initially specified, the coordinate boundary points should have the same (u, v) values as obtained from (1) for the specified boundaries. The corresponding positions of the points on the opposite boundaries in the physical (x, y) domain are now obtained by using Eq. (6). These points also have the same (ξ, η) values as the boundary points for which the boundary correspondence was originally specified.

(s3) Now, we know $u(\xi)$ and $v(\eta)$ on all boundaries, hence we can calculate $u'(\xi)$ and $v'(\eta)$.

Then, for a map of the assumed form, we can calculate

$$f(\xi, \eta) = \frac{v'(\eta)}{u'(\xi)}. \tag{9}$$

Hence, we have obtained all necessary informations, namely $f(\xi, \eta)$ and the previously unspecified boundary correspondence on the *opposite* boundaries, so that the mapping problem now involves the linear equations (Eq. (1)) with f given, which are to be solved subject to Dirichlet boundary conditions on all boundaries.

3.2. Step 2: Solving the Covariant Laplace Equations

From the previous step, we have determined all informations needed to obtain a unique solution of the covariant Laplace equations (Eqs. (1a) and (1b)). Since the equations with specified $f(\xi, \eta)$ are linear and the boundary conditions are of the Dirichlet type, it is simple to solve the mapping problem by a direct method. In the next section, we present the details of the boundary integral formulation for the first step in the mapping scheme.

4. BOUNDARY INTEGRAL FORMULATION OF THE CONFORMAL MAPPING PROBLEM

As mentioned in the previous section, there is one parameter of the conformal map that must be determined as part of the solution, namely v^*. Thus, we begin by solving the u-problem first in order to determine v^* by Eq. (4). Since the two problems for u and v are identical except for the boundary conditions, we limit our discussion to the u-problem only. The objective, as with other applications of the BI method, is to use the appropriate form of Green's formula and fundamental solutions of the governing equation to transform the problem (Eq. (3a)) to an integral formula which specifically gives the data that is unknown on each segment of the boundary in terms of the data that has been

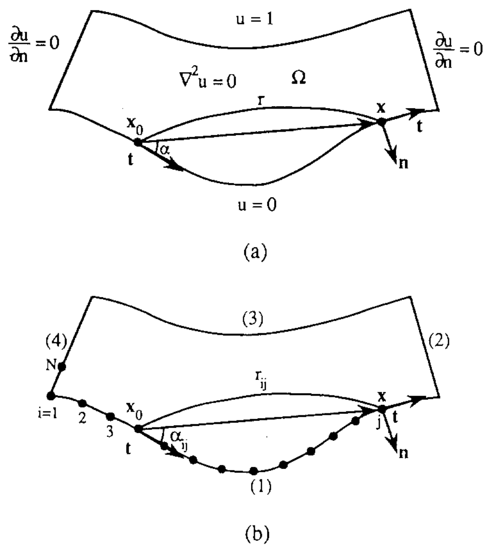


FIG. 5. An illustration for the BI method: (a) notations for integral equation; (b) discretization along the boundary.

specified. For example, the values of u are unknown on segments 2 and 4, while $\partial u/\partial n$ are unknown on segments 1 and 3.

Let us consider an arbitrary simply connected 2D domain as shown in Fig. 5. Green's formula for a harmonic function in the 2D domain is given by

$$\beta u(\mathbf{x}) = \int_{\partial\Omega} G(\mathbf{x}-\mathbf{x}_0)\left(\frac{\partial u}{\partial n}\right) dt - \int_{\partial\Omega} u\left(\frac{\partial G}{\partial n}\right) dt, \qquad (10)$$

where $\beta = 2\pi$ if $\mathbf{x} \in \Omega$, $\beta = \pi$ if $\mathbf{x}$ is on the smooth part of the boundary, and $\beta = \theta$ if $\mathbf{x}$ is on a corner of the boundary, where θ is the angle of the corner. The Green's function for the 2D problem is

$$G(\mathbf{x}-\mathbf{x}_0) = -\log|\mathbf{x}-\mathbf{x}_0| = -\log r,$$

and it is easily shown that

$$\frac{\partial G}{\partial n} = -\frac{d\alpha}{dt},$$

where α is the angle between the vector $\mathbf{x}_0 - \mathbf{x}$ and the tangent to an appropriate cut starting from $\mathbf{x}$ but not passing through the given domain D (see Fig. 5). Thus, substituting for G and $\partial G/\partial n$ in (10), we obtain an integral formula for u in terms of boundary values of u and $\partial u/\partial n$.

$$\beta u = -\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)\log r\, dt + P\int_{\partial\Omega} u\frac{d\alpha}{dt}\, dt. \qquad (11)$$

Alternatively, using the relation

$$P\int_{\partial\Omega} u\frac{d\alpha}{dt}\, dt = \beta u - P\int_{\partial\Omega}\alpha\frac{du}{dt}\, dt,$$

we have

$$P\int_{\partial\Omega}\alpha\frac{du}{dt}\, dt + \int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)\log r\, dt = 0. \qquad (12)$$

Now, if we define $w = \partial u/\partial n$, and represent u and w using a finite number of nodal values, then (12) can be represented by the following vector equation:

$$\mathbf{A}\mathbf{u} = \mathbf{G}\mathbf{w}. \qquad (13)$$

Here $\mathbf{A} = (\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3, \mathbf{A}^4)$, $\mathbf{G} = (\mathbf{G}^1, \mathbf{G}^2, \mathbf{G}^3, \mathbf{G}^4)$, $\mathbf{u} = (\tilde{\mathbf{u}}_1, \mathbf{u}_2, \tilde{\mathbf{u}}_3, \mathbf{u}_4)^{\mathrm{T}}$, and $\mathbf{w} = (\mathbf{w}_1, \tilde{\mathbf{w}}_2, \mathbf{w}_3, \tilde{\mathbf{w}}_4)^{\mathrm{T}}$, where the subscripts and the superscripts denote the boundary label (i.e., sides 1–4 in the transform domain), and the tilde is used to indicate the known variables on the particular boundary segment (i.e., u is specified on sides 1 and 3, while w is specified on sides 2 and 4). By rearranging (13) we have

$$\mathbf{M}\mathbf{x} = \mathbf{N}\tilde{\mathbf{x}}, \qquad (14)$$

where $\mathbf{M} = (-\mathbf{G}^1, \mathbf{A}^2, -\mathbf{G}^3, \mathbf{A}^4)$, $\mathbf{N} = (-\mathbf{A}^1, \mathbf{G}^2, -\mathbf{A}^3, \mathbf{G}^4)$, $\mathbf{x} = (\mathbf{w}_1, \mathbf{u}_2, \mathbf{w}_3, \mathbf{u}_4)^{\mathrm{T}}$, and $\tilde{\mathbf{x}} = (\tilde{\mathbf{w}}_1, \tilde{\mathbf{u}}_2, \tilde{\mathbf{w}}_3, \tilde{\mathbf{u}}_4)^{\mathrm{T}}$. In (14), the right hand side is known, so the solution is given by

$$\mathbf{x} = \mathbf{M}^{-1}(\mathbf{N}\tilde{\mathbf{x}}). \qquad (15)$$

The formula (15) provides an explicit basis to calculate the values of u and w on the boundary that have not been specified in the formulation of the mapping problem.

5. EXAMPLES OF APPLICATION

We have applied the scheme developed in the previous sections to generate grid systems for several geometries that are of interest in fluid mechanics. For illustration purposes, we have considered two specific geometries, the first consisting of two cylinders in a big square box, and the second corresponding to the region between two big cylinders as shown in Figs. 6 and 7. Due to the symmetries present in the problem, we have only considered one quarter of the whole geometry in Fig. 6 and one half in Fig. 7.

The grid system in Fig. 6 was generated using a 41×11 grid with the specified distortion function $f(\xi, \eta) = 1 - 0.8\xi$. The specific distortion function was chosen to avoid a very dense grid in the region between cylinders which would be

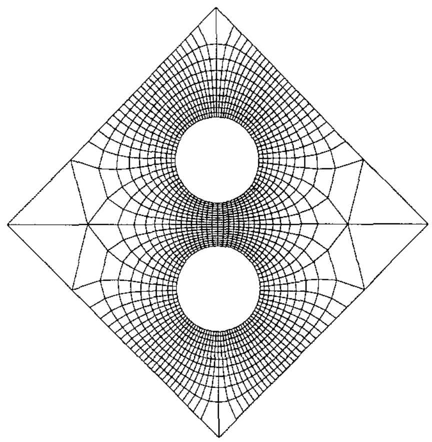


FIG. 6. Orthogonal mapping with prescribed $f(\xi, \eta) = 1 - 0.8\xi$.

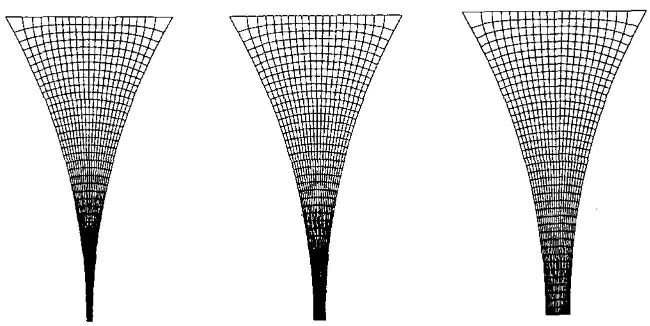


FIG. 7. Orthogonal mapping with prescribed boundary correspondence on two adjacent sides.

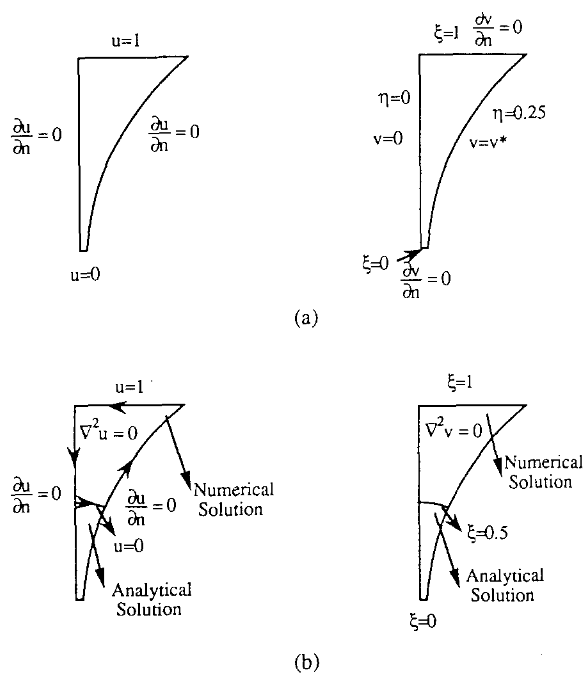


FIG. 8. A domain with almost singular region: (a) direct application of BI method; (b) composition of analytical and numerical solutions.

obtained if the conformal mapping ($f = 1$) were used. With $f(\xi, \eta) = 1 - 0.8\xi$, we have determined $u(\xi)$, $v(\eta)$ as

$$u(\xi) = -\ln(1 - 0.8\xi)/\ln 5, \tag{16}$$

$$v(\eta) = (\ln 5/0.8)\eta. \tag{17}$$

From the boundary integral solution for the u-problem, we found $v^* = 0.246$ via Eq. (4), and determined η^* by the relation $v(\eta^*) = v^*$. The step sizes of the Cartesian grid in the (ξ, η)-domain (computational domain) were then determined as $\Delta\xi = \frac{1}{40}$, $\Delta\eta = \eta^*/10$. With these step sizes and Eqs. (16) and (17), u and v values corresponding to the boundary points were obtained to produce the Dirichlet-type boundary conditions for Eq. (1), according to the procedure shown below Eq. (8).

For the 41×11 grid systems in Fig. 7, we have specified boundary correspondences at the center ($\eta = 0$) and the bottom ($\xi = 0$) lines (see also Fig. 8). On each of those lines, the boundary points are equally spaced. Grid generation for this problem was carried out in a rather special way as shown below. As we may see, the geometry in this problem is quite unusual in the sense that one side (bottom side) is much shorter than others; i.e., the geometry includes an almost singular region. Therefore, if we apply our scheme to this problem without any treatment as shown in Fig. (8a), there may be considerable error in the boundary integral solution near the singular region due to the singular nature of the harmonic function near the singular point. To avoid the inherent numerical error, we decomposed the domain into two parts as shown in Fig. (8b). For the singular region ($0 \leqslant \xi \leqslant 0.5$), we have generated an analytical grid by modifying the bipolar coordinate system, from which we also determined the position of grid points on the coordinate line corresponding to $\xi = 0.5$. After this step, the remaining problem was grid generation for the regular region corresponding to $0.5 \leqslant \xi \leqslant 1$, with specified boundary correspondences at adjacent boundaries, which was performed successfully as shown in Fig. 7.

As we can see in Figs. 6 and 7, the orthogonality is excellent everywhere except for the vicinity of the nonorthogonal corner points. The major source of error is believed to come from the coarse grid distribution near that point, which is an intrinsic property of a single grid system for such a peculiar geometry. However, fortunately the coarse grid system near that point will not result in any significant error in the fluid mechanics problem because that region is a stagnation region for the flow. One more point to be stressed here is that our scheme is also very effective for the generation of composite grid systems. Geometries in some problems are so complicated that we may need to divide the whole domain into several subdomains to generate a complete grid system by combining smaller grid systems for subdomains. In that case, two adjacent subdomains should have common boundary points where the grid lines should also be continuous. As explained above for Fig. 7, our scheme for the case of specified boundary correspondence will be a powerful tool for those problems.

6. SIMPLIFICATION OF INTRINSICALLY NONLINEAR MAPPING PROBLEMS

So far we have limited our discussion of orthogonal grid generation methods to special cases in which the generating equations (Eqs. (1a) and (1b)), plus boundary conditions, can be reduced via a preliminary conformal step to a linear system. This is because our primary interest is given to mappings that can be solved by a direct, noniterative solution scheme. For more general classes of problems such as construction of an orthogonal grid system with specified boundary correspondence on *all* boundaries, the problem is intrinsically nonlinear and the scheme must be iterative. Indeed, several different iterative grid generation schemes have already been proposed for this more general class of problems (e.g., the weak constraint method of Ryskin and Leal). Unfortunately, a general proof of existence or uniqueness of a solution for this type of mapping problem is not available. Even if we take the existence of an orthogonal map for granted, however, it can still be very difficult in practice to obtain a converged solution to the mapping problem. One reason is that the generation of an initial guess can be difficult if the shape of the physical domain is complicated. It follows, then, that an important simplification may occur if the boundary-integral technique that was used as the initial step in the "linear" grid generation scheme is also applied to this more general class of mapping problems. The basic idea is to use the BI-based conformal mapping technique to transform the problem from the complicated, physical domain to a rectangular domain, so that the iterative orthogonal mapping step can be done in the simpler framework of transferring from one rectangular domain to another (see Fig. 9).

Our discussion should start with the notion that any orthogonal mapping between a given 2D domain and some rectangular domain (denoted as T_o: $(\xi, \eta) \in \Omega_\xi \to (x, y) \in \Omega_x$) can be decomposed into a conformal mapping (denoted as T_c: $(u, v) \in \Omega_u \to (x, y) \in \Omega_x$) followed by an orthogonal mapping (denoted as T_{o_1}: $(\xi, \eta) \in \Omega_\xi \to (u, v) \in \Omega_u$) as show in Fig. 9. More formally, the decomposition can be written as

$$T_o = T_c T_{o_1}. \tag{18}$$

Now, if we denote the distortion factors for the mappings T_o, T_c, and T_{o_1} as f, f_c, and f_1, respectively, then it follows from the property of conformal mapping, $f_c = 1$, that the

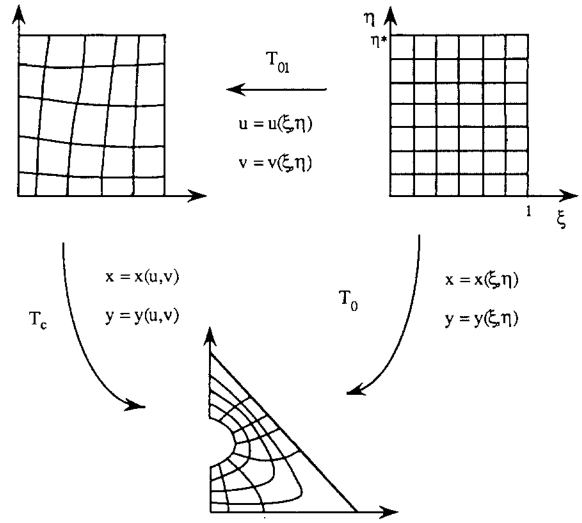


FIG. 9. Simplification of difficult nonlinear problem (decomposition of an orthogonal mapping into a conformal mapping and an orthogonal mapping in a simple geometry).

distortion factor for the complete map is the same as that for the orthogonal step in (18), i.e.,

$$f(\xi, \eta) = f_1(\xi, \eta) \qquad \forall (\xi, \eta) \in \Omega_\xi. \tag{19}$$

The implication of (19) is that in order to determine the distortion function for the overall mapping, f, it is sufficient to consider the simplified mapping problem T_{o_1}. Now, the question is how to generate the simplified problem from the formulation of the original problem, and this is discussed below.

Here we consider the problem for which the boundary correspondence is fully specified at all boundary points. This general class of problems can be treated according to the following steps.

Step 1. By the method discussed in Step 1 of Section 3, map the specified boundary correspondence on the physical domain $\partial\Omega_x$ to the corresponding boundary points on the rectangular domain $\partial\Omega_u$ (i.e., find $(u(\xi, \eta), v(\xi, \eta)) \in \partial\Omega_u$ corresponding to each specified point $(x(\xi, \eta), y(\xi, \eta)) \in \partial\Omega_x$). Thus, in effect, we transfer from the original problem to a rectangular domain with specified boundary correspondence at all points.

By Step 1 we therefore reduce the problem of mapping an arbitrary 2D domain to *the problem of orthogonal grid generation from one rectangular domain to another with specified boundary correspondences.*

Step 2. Generate an orthogonal coordinate system in the rectangular domain using some iterative scheme. During this nonlinear process $f_1(\xi, \eta)$ is determined. Since $f(\xi, \eta) = f_1(\xi, \eta)$, we now have $f(\xi, \eta)$ for the overall mapping problem.

Step 3. The basic problem now is to transform from the orthogonal grid in the transformed domain back to corresponding grid lines in the physical domain. For this, we need to solve the mapping equations (Eqs. (1a) and (1b)). However, since $f(\xi, \eta) = f_1(\xi, \eta)$ is known, the grid generating equation is now linear. Furthermore, we already have Dirichlet-type boundary conditions at the boundaries from the specified boundary correspondences. Therefore, the final step in the mapping problem is to solve two linear pde's with Dirichlet boundary conditions.

Here we do not present any examples of application because we need to discuss the iterative solution method for the nonlinear problem in Step 2. However, we hope that this idea can be useful to other investigators.

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REFERENCES

1. J. F. Thompson, Z. U. A. Warsi, and C. W. Mastin, *Numerical Grid Generation* (North-Holland, New York, 1985).
2. B. Fornberg, *SIAM J. Sci. Stat. Comput.* **1**, 386 (1980).
3. G. T. Symm, *Numer. Math.* **9**, 250 (1966).
4. R. Menikoff and C. Zemach, *J. Comput. Phys.* **36**, 366 (1980).
5. C. D. Mobley and R. J. Stewart, *J. Comput. Phys.* **34**, 124 (1980).
6. G. Ryskin and L. G. Leal, *J. Comput. Phys.* **50**, 71 (1983).